THE INSTABILITY OF TRANSVERSE WAVES IN A RELATIVISTIC PLASMA

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ABSTRACT

Linearized equations are set up to describe disturbances in an infinite, spatially uniform, relativistic plasma without an ambient magnetic field. It is shown that, as well as the usual electrostatic waves, there also exists a class of electromagnetic waves. The two sets of waves are coupled in general, but can still be classified as mainly longitudinal or mainly transverse. Under the assumption that the system is stable against the longitudinal disturbances it is shown that the relativistic plasma will be unstable to the transverse waves unless it is virtually isotropic.

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1. Introduction

The average age of cosmic rays in the galactic disc is estimated to be of the order of 5.10^6 years. Also it is known from observation that cosmic rays are isotropic to better than 1% (Greisen, 1956).

These two facts, together with the supernovae theory of the origin of cosmic rays, make it important to find a mechanism which will reduce an arbitrary amount of anisotropy (since presumably supernovae produce cosmic rays anisotropically) to less than about 1% in a time less than, or of the order of, the mean cosmic ray lifetime.

It has been conjectured that interstellar magnetic field irregularities will produce some measure of isotropy due to pitch angle scattering. However, not much is known about the scale size of such irregularities.

It is therefore of interest to examine other possible ways of producing some degree of isotropy in an initially anisotropic relativistic plasma. One such possibility is particle velocity redistribution due to the influence of plasma waves. It is well known that it is difficult to make plasma waves carry a significant amount of energy but for producing isotropy this is not a prime requirement. In fact, the plasma waves need only re-order the plasma distribution function in order to achieve some measure of isotropy. In this sense the waves take on the role of collisions in a classical gas.

In discussing the behavior of a plasma perturbed by a disturbance, attention is usually restricted to purely electrostatic waves since these grow in the order of a plasma period which, in the absence of an ambient magnetic field, is the shortest possible time for a dynamical process.

However there also exists a class of electromagnetic disturbances whose existence has been recognized by several authors (Fried, 1959; Weibel, 1959). Such electromagnetic waves have been considered in considerable detail for a non-relativistic plasma (Kahn, 1962).

The main reason for considering such waves is essentially due to the conditions attached to making the plasma unstable against the electrostatic mode. These conditions are well known (Penrose, 1960). It has been shown (Noerdlinger, 1961) that the requirements for electrostatic instability are difficult to meet in several interesting astrophysical situations.

It is therefore worthwhile considering the electromagnetic waves since the requirements for instability of these waves are much easier to meet. It should be emphasized that these waves are not the familiar fast electromagnetic waves with phase velocities of the order of c, the velocity of light. In fact if the r.m.s. velocity spread is σc the electromagnetic waves to be considered generally have amplification rates of the order of σ times the plasma frequency. Consequently, they are not nearly as violent as the electrostatic waves. They do have the advantage that the plasma need not obey such stringent requirements as the electrostatic mode demands before they become unstable.

In this analysis we do not allow for a galactic magnetic field despite the observational evidence which indicates the existence of such a field with a mean strength of about 5×10^{-6} Γ (Gardner and Davies, 1965). The plethora of complications which arise when an ambient magnetic field is taken into account have been the subject of innumerable papers and books and we make no attempt to consider them.

In several recent papers (Lerche, 1965a, b, 1966) particular attention was paid to the electromagnetic and space charge waves when the relativistic plasma was embedded in an infinite, homogeneous magnetic field. In all these papers the tacit assumption was made that there was no coupling between the two types of wave. We will demonstrate in this paper that, in the absence of an ambient magnetic field, coupling exists but barely influences the space charge wave. It will also be shown that the coupling seriously perturbs the electromagnetic wave. Thus the assumption of no coupling in an ambient magnetic field is suspect and should be investigated. In particular under the assumption of no coupling it can be shown (Lerche, 1965b) that the electromagnetic mode, in an ambient magnetic field, does not grow at a physically significant rate for the cosmic ray gas. This result may not be true when interference is allowed for.

Thus this paper cannot describe the behavior of the galactic cosmic ray gas in the general galactic magnetic field. The motivation behind this work is essentially self-educative. We hope that the results presented here lead to a better understanding of the physical behavior of relativistic plasmas.

As remarked earlier, a similar calculation to the following has been performed for a non-relativistic plasma (Kahn, 1962). It is not immediately obvious that Kahn's criteria for stability against the electromagnetic waves can be applied to a relativistic plasma. We will show that while the physical sense of Kahn's criteria is preserved the mathematical formalism changes due to the relativistic nature of the problem.

We will make no attempt to calculate instability rates for the unstable situations. Such a calculation would require a detailed knowledge of the distribution function and in this paper we shall only be concerned with general properties that a distribution function must possess in order to avoid instability.

Further, since we do not include an ambient magnetic field in the calculations, even if we were to calculate e-folding times for particular distribution functions, we could not place any reliance on them as measures of the speed with which an instability occurs in the galactic cosmic ray gas.

2. The Dispersion Relation

We consider only the case of a mobile relativistic proton plasma without an ambient magnetic field. It suffices to consider one mobile species since the theory to be developed uses linearized equations. Thus more than one mobile species can easily be taken into account. Along with the proton plasma we assume that there exists a cold, smeared out, electron charge background which does not contribute to the motion and serves to preserve over-all space charge neutrality.

We let the equilibrium relativistic proton distribution function be f_0 and the first order linear perturbation to f_0 be f_1. It is then a simple matter to show that f_1 satisfies the linearized relativistic Vlasov equation

$$\frac{\partial f_1}{\partial t} + \frac{c\,\underset{\sim}{p}}{\sqrt{(1+p^2)}} \cdot \frac{\partial f_1}{\partial \underset{\sim}{x}} + \frac{\epsilon}{mc}\left[-\nabla\varphi - c^{-1}\frac{\partial \underset{\sim}{A}}{\partial t} + \frac{\underset{\sim}{p}\times(\nabla\times\underset{\sim}{A})}{\sqrt{(1+p^2)}}\right]\cdot\frac{\partial f_0}{\partial \underset{\sim}{p}}$$

$$= 0 , \qquad (1)$$

where ϵ and m are the charge and rest mass of a proton. The normalized momentum, $\underset{\sim}{p}$, is defined in terms of the real momentum, $\underset{\sim}{P}$, through the relation $mc\underset{\sim}{p} = \underset{\sim}{P}$. Here φ and $\underset{\sim}{A}$ are the scaler electrostatic and vector electromagnetic potentials respectively.

We must also satisfy the Maxwell equations

$$\nabla^2\varphi - c^{-2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\epsilon\int f_1\, d^3p \quad , \tag{2}$$

$$\nabla^2\underset{\sim}{A} - c^{-2}\frac{\partial^2\underset{\sim}{A}}{\partial t^2} = -4\pi\epsilon\int\frac{f_1\,\underset{\sim}{p}\,d^3p}{\sqrt{(1+p^2)}} \quad . \tag{3}$$

In addition we must ensure the preservation of the gauge condition

$$\nabla\cdot\underset{\sim}{A} + c^{-1}\frac{\partial\varphi}{\partial t} = 0 \quad . \tag{4}$$

We choose a particular Cartesian coordinate system and allow all first order perturbation quantities to vary as

$$\exp\left[ik(x-c\beta t)\right] \quad . \tag{5}$$

We let k be real and positive and β complex without any loss of generality.

Making use of (5) it can easily be shown that the solution to (1) is given by

$$f_1 = \frac{\epsilon}{mc^2}\left(A_y \frac{\partial f_0}{\partial p_y} + A_z \frac{\partial f_0}{\partial p_z}\right) + \frac{\epsilon\left[\varphi(1-\beta^2) - \frac{(p_y A_y + p_z A_z)}{\sqrt{(1+p^2)}}\right]}{mc^2\left[p_x/\sqrt{(1+p^2)} - \beta\right]}\frac{\partial f_0}{\partial p_x} \,, \qquad (6)$$

where use has been made of the gauge condition in the form

$$A_x = \beta\varphi$$

We now normalize f_0 so that

$$\int f_0 \, d^3p = 1 \,, \qquad (7)$$

when it can be shown that, with the aid of (5), (2) becomes

$$k^2(1-\beta^2)\varphi = \frac{4\pi\epsilon^2 N}{mc^2}\int dp_x \, dp_y \, dp_z \; \times$$

$$\left\{A_y \frac{\partial f_0}{\partial p_y} + A_z \frac{\partial f_0}{\partial p_z} + \frac{\left[\varphi(1-\beta^2) - (p_y A_y + p_z A_z)(1+p^2)^{-1/2}\right]}{\left[p_x(1+p^2)^{-1/2} - \beta\right]}\frac{\partial f_0}{\partial p_x}\right\} , \qquad (8)$$

and the range of integration in (7) and (8) is

$$-\infty \leq p_x \leq \infty \ ; \quad -\infty \leq p_y \leq \infty \ ; \quad -\infty \leq p_z \leq \infty \ .$$

With the help of (5) is may also be shown that (3) can be written

$$k^2(1-\beta^2)A_y = \frac{4\pi\epsilon^2 N}{mc^2}\int \frac{p_y}{\sqrt{(1+p^2)}} \{ \quad \} \, dp_x dp_y dp_z \ , \tag{9}$$

$$k^2(1-\beta^2)A_z = \frac{4\pi\epsilon^2 N}{mc^2}\int \frac{p_z}{\sqrt{(1+p^2)}} \{ \quad \} \, dp_x dp_y dp_z \ , \tag{10}$$

where the parentheses in (9) and (10) denote the factor in paren- theses occuring in (8). Here N is the number density of relativistic particles.

Defining

$$\Phi = \varphi(1-\beta^2) \ , \quad k_0^2 = 4\pi\epsilon^2 N/(mc^2) \ , \qquad \kappa^2 = k^2 k_0^{-2}$$

and noting that (8), (9), and (10) are all linear in Φ, A_y and A_z, we see that for a solution to exist to these three equations we must demand that

$$\begin{vmatrix} \kappa^2 - I_1(\beta) & I_y(\beta) & I_z(\beta) \\ -I_y(\beta) & \kappa^2(1-\beta^2) + g_x + g_z + I_{yy}(\beta) & I_{yz}(\beta) - h_{yz} \\ -I_z(\beta) & I_{yz}(\beta) - h_{yz} & \kappa^2(1-\beta^2) + g_x + g_y + I_{zz}(\beta) \end{vmatrix} = 0, \tag{11}$$

where

$$g_x = \int \frac{f_0\,(1+p_x^2)\,d^3p}{(1+p^2)^{3/2}} \;; \quad (g_y, g_z) = \int \frac{f_0\,(p_y^2, p_z^2)\,d^3p}{(1+p^2)^{3/2}} \;,$$

$$h_{yz} = \int \frac{f_0\, p_y\, p_z\, d^3p}{(1+p^2)^{3/2}} \;; \quad I_1(\beta) = \int \frac{\sqrt{(1+p^2)}\;\partial f_0/\partial p_x\; d^3p}{[p_x - \beta\sqrt{(1+p^2)}\,]} \;,$$

$$(I_y, I_z, I_{yz}, I_{yy}, I_{zz}) = \int \frac{f_0\, d^3p}{[p_x - \beta\sqrt{(1+p^2)}]} \times \left(p_y, p_z, \frac{p_y p_z}{\sqrt{(1+p^2)}}, \frac{p_y^2}{\sqrt{(1+p^2)}}, \frac{p_z^2}{\sqrt{(1+p^2)}}\right) .$$

We have also assumed that f_0 satisfies the usual convergence conditions as p_x, p_y or $p_z \to \pm\infty$.

From the dispersion equation (11) we have a relation between k and β. An unstable situation will develop if, and only if, a real, positive k exists for which β has a positive imaginary part.

3. Some Aspects of the Dispersion Relation.

One point, which is immediately obvious from (11), is that if β is real, positive and greater than unity all the integrals in (11) are completely real since, for all values of p_x, p_y and p_z, we have $p_x < \beta\sqrt{(1+p_x^2+p_y^2+p_z^2)}$. This merely states that waves with phase velocities greater than c do not resonate with the finite rest mass protons which always have sub-luminous velocities.

Also it can readily be shown that, provided f_0 is not pathological, all the integrals in (11) are analytic functions of β in any one half plane. Since we are looking for temporal instability we choose to define k real and positive and β in the upper half plane. It is then well known that as $\mathrm{Im}(\beta) \to 0+$ from above the resulting functions of β are also analytic on the real β axis (Jackson, 1958).

Suppose we now choose the zero velocity to the mean particle velocity, say. Then if the equilibrium distribution has a mean velocity half width σc we see that when $\beta >> \sigma$ we have

$$I_1(\beta) = O(\beta^{-2}).$$

Likewise

$$I_y(\beta) = O(\sigma\beta^{-2}) ,$$

and so on.

For those waves with phase velocities close to c we have $\sigma \ll |\beta|$ in most physical situations. Neglecting terms of order $\sigma^2\beta^{-2}$ we see that in such a case (11) reduces to just its diagonal elements, and the electrostatic and electromagnetic waves completely decouple. We then obtain the usual relation

$$\kappa^2 - I_1(\beta) = 0 , \tag{12}$$

for the electrostatic mode. The corresponding relation for the electromagnetic mode is

$$k^2\beta^2 = k^2 + k_0^2 . \tag{13}$$

However we are interested in the situation where $|\beta| \ll 1$. In this case we can replace the factor $(1-\beta^2)$ by unity in the electromagnetic diagonal terms of (11). Then the dispersion relation becomes

$$\begin{vmatrix} \kappa^2 - I_1(\beta) & I_y(\beta) & I_z(\beta) \\ -I_y(\beta) & \kappa^2 + g_x + g_z + I_{yy}(\beta) & I_{yz}(\beta) - h_{yz} \\ -I_z(\beta) & I_{yz}(\beta) - h_{yz} & \kappa^2 + g_x + g_y + I_{zz}(\beta) \end{vmatrix} = 0 . \tag{14}$$

It may happen that a situation is chosen with sufficient symmetry so that

$$I_y(\beta) = 0 = I_z(\beta) . \tag{15}$$

In this case the electrostatic and electromagnetic modes completely decouple. In general, however, the integrals in (15) will not vanish and they introduce cross-coupling between the two different types of modes. From (12) we see that the order of magnitude calculation shows that we are predominantly interested in those electrostatic modes for which

$$\kappa^2 = O(\beta^{-2}) \quad , \quad |\beta| >> \sigma$$
$$\kappa^2 = O(\sigma^{-2}) \quad , \quad |\beta| << \sigma .$$

In the present situation we are looking at $|\beta| << 1$ and in particular we will assume that $\sigma >> |\beta|$. Thus we expect κ^2 to be of the order σ^{-2} for the electrostatic mode. This is much larger than $\kappa^2 = O(1)$ which we expect for the electromagnetic mode. Thus as far as the electrostatic mode is concerned the coupling can be represented to a good enough approximation by

$$\begin{vmatrix} \kappa^2 - I_1(\beta) & I_Y(\beta) & I_z(\beta) \\ -I_Y(\beta) & \kappa^2 & 0 \\ -I_z(\beta) & 0 & \kappa^2 \end{vmatrix} = 0 . \tag{16}$$

Setting

$$\kappa^2 = I_1(\beta)$$

in the second and third diagonal terms, which is accurate to the order required, we see that (16) becomes

$$\kappa^2 - I_1(\beta) \simeq - I_1^{-1}(\beta) \left[I_Y^2(\beta) + I_Z^2(\beta) \right] .$$

(17)

Thus an extra term of order σ^2 times the dominant term has been introduced. This hardly affects the electrostatic made at all and consequently the usual electrostatic dispersion equation is a good enough approximation to the correct relation.

However the coupling of the electrostatic mode to the electromagnetic waves is not negligible. For the transverse waves, we are interested in values of κ^2 of the order of unity while

$$I_1(\beta) = O(\sigma^{-2})$$

for the slow electrostatic wave. Thus as far as the transverse wave is concerned a good enough approximation to the dispersion relation is

$$\begin{vmatrix} I_1(\beta) & I_Y(\beta) & I_Z(\beta) \\ I_Y(\beta) & \kappa^2+g_x+g_z+I_{YY}(\beta) & I_{YZ}(\beta)-h_{YZ} \\ I_Z(\beta) & I_{YZ}(\beta)-h_{YZ} & \kappa^2+g_x+g_y+I_{ZZ}(\beta) \end{vmatrix} = 0 . \tag{18}$$

It is then a simple matter to show that (18) can be written

$$\left[q+\frac{1}{2}\left(\mathcal{J}_{YY}+\mathcal{J}_{ZZ}\right)\right]^2=\frac{1}{4}\left(\mathcal{J}_{YY}-\mathcal{J}_{ZZ}\right)^2+\mathcal{J}_{YZ}^2 , \tag{19}$$

where

$$q=\kappa^2+g_x ; \quad \mathcal{J}_{YY}=g_z+I_{YY}-I_Y^2 I_1^{-1} ;$$

$$\mathcal{J}_{ZZ}=g_y+I_{ZZ}-I_Z^2 I_1^{-1} ; \quad \mathcal{J}_{YZ}=I_{YZ}-h_{YZ}-I_Y I_Z I_1^{-1} .$$

4. Stability Considerations.

As has been done in the non-relativistic case (Kahn, 1962) we will now demonstrate that the class of electromagnetic waves whose dispersion relation is given by (19) is unstable unless the equilibrium relativistic proton distribution function satisfies some rather restrictive conditions.

We shall consider only the case of even parity distribution functions, i.e.,

$$f_0(p_x, p_y, p_z) = f_0(-p_x, -p_y, -p_z) \tag{20}$$

Making use of (20) it can easily be shown that I_1, I_{yy}, I_{zz}, I_{yz}, are real and I_y, I_z, are pure imaginary when β is pure imaginary. We note also that g_x, g_y, and g_z are real and positive.

A sufficient condition for electromagnetic instability is that there exist a real, positive k whose β has a positive, imaginary part. Since the right hand side of (20) is real and positive and since $q = g_x + k^2 k_0^{-2}$, this means that there exists a β in the upper half complex plane whose corresponding q is real and greater than g_x. This is so if $J_{yy} + J_{zz}$ is real and less than $-2g_x$ somewhere on the imaginary β axis in the upper half plane. Hence, by continuity, if

$$J_{yy} + J_{zz} < -2g_x \tag{21}$$

when $\beta = 0$.

We can also ignore the class of situation for which I_1 is real and positive anywhere in the upper half complex β plane, since it follows that the plasma will then be unstable against electrostatic waves. These will dominate over the slow electromagnetic waves.

Thus the physically interesting situation is that in which I_1 and $I_y^2 + I_z^2$ are negative on $\beta = i\xi \ (\xi > 0)$.

On $\beta = i\xi$ we therefore have

$$\mathcal{J}_{yy} + \mathcal{J}_{zz} = g_y + g_z + I_{yy} + I_{zz} - \left(I_y^2 + I_z^2\right) I_1^{-1} < -2g_x ,$$

and

$$\mathcal{J}_{yy} + \mathcal{J}_{zz} \leqslant g_y + g_z + I_{yy} + I_{zz} < -2g_x .$$

To avoid instability we require that

$$I_{yy}(0) + I_{zz}(0) \geqslant -2g_x - g_y - g_z , \tag{22}$$

with equality if, and only if, $I_y(0) = 0 = I_z(0)$.

Now

$$I_{yy}(0) + I_{zz}(0) = \int \frac{(p_y^2 + p_z^2)}{p_x\sqrt{(1+p^2)}} \frac{\partial f_0}{\partial p_x} d^3p \equiv M(0), \text{ say.} \tag{23}$$

We now change to spherical momentum coordinates defined by

$$p_x = p\cos\theta\ , \quad p_y = p\sin\theta\cos\varphi\ , \quad p_z = p\sin\theta\sin\varphi$$

so that

$$\frac{\partial}{\partial p_x} = \cos\theta\frac{\partial}{\partial p} - \frac{\sin\theta}{p}\frac{\partial}{\partial\theta}\ .$$

Setting

$$\int_0^{2\pi} f_0(p,\theta,\varphi)\,d\varphi = 2\pi F(p,\theta)\ ,$$

we see that (23) becomes

$$M(0) = 2\pi\int_0^\infty \frac{p^2\,dp}{\sqrt{(1+p^2)}}\int_{-1}^{+1}(1-\mu^2)\left[p\frac{\partial F}{\partial p} + \mu^{-1}(1-\mu^2)\frac{\partial F}{\partial\mu}\right]d\mu\ , \tag{24}$$

where $\mu = \cos\theta$.

We note that

$$\int_0^\infty \frac{p^3}{\sqrt{(1+p^2)}}\frac{\partial F}{\partial p}\,dp = -3\int_0^\infty \frac{p^2 F\,dp}{\sqrt{(1+p^2)}} + \int_0^\infty \frac{p^4 F\,dp}{(1+p^2)^{3/2}}\ .$$

We now expand $F(p, \theta)$ in terms of Legendre polynomials

$$F(p, \theta) = \sum_{n=0}^{\infty} R_{2n}(p) \, P_{2n}(\mu) \quad , \tag{25}$$

where the assumption of even parity ensures that only even polynomials in μ enter (25).

Setting

$$r_{2n} = \int_0^{\infty} \frac{p^2 R_{2n}(p) \, dp}{\sqrt{(1+p^2)}}$$

and

$$J_{2n} = \int_0^{\infty} \frac{p^4 R_{2n}(p) \, dp}{(1+p^2)^{3/2}} \quad ,$$

we see that (24) can be written

$$M(0) = 2\pi \sum_{n=0}^{\infty} r_{2n} \int_{-1}^{+1} \left[-3(1-\mu^2) P_{2n}(\mu) + \mu^{-1}(1-\mu^2)^2 P'_{2n}(\mu) \right] d\mu$$

$$+ 2\pi \sum_{n=0}^{\infty} J_{2n} \int_{-1}^{+1} (1-\mu^2) P_{2n}(\mu) \, d\mu \quad , \tag{26}$$

where the prime denotes differentiation with respect to the argument.

It can be shown (Kahn, 1962) that

$$\int_{-1}^{+1}\left[-3(1-\mu^2)P_{2n}(\mu)+\mu^{-1}(1-\mu^2)^2P_{2n}'(\mu)\right]d\mu \equiv i_n, \text{ say}$$

$$= -4 \,, \quad n=0$$

$$= \frac{(-1)^{n-1}\,2^{2n+1}\,(n!)^2}{(2n)!} \,, \quad n \geqslant 1$$

so that

$$M(0) = -8\pi r_0 + 2\pi\sum_{n=1}^{\infty} i_n r_{2n} + 8\pi/3\left(J_0 - J_2/5\right). \tag{27}$$

It can easily be shown that

$$g_y + g_z = 8\pi/3\left(J_0 - J_2/5\right)$$

and

$$g_x + g_y + g_z = 4\pi r_0 \quad .$$

Thus the condition that instability be avoided can be written

$$\sum_{n=1}^{\infty} i_n r_{2n} \geqslant 0 \quad . \tag{28}$$

To avoid instability this result must hald true not only for the one particular direction of the wave chosen, namely along the x-axis, but for <u>any</u> direction of the wave normal.

We therefore define a basic direction with respect to which a given wave normal points into the direction (λ_0, ν_0). With respect to this basic direction we can write

$$f_0(\underline{p}) = f_0(p, \lambda, \nu) = \Psi_0(p) + \sum_{n=1}^{\infty} \sum_{m=0}^{2n} \Psi_{2n}^{(m)}(p) S_{2n}^{(m)}(\lambda, \nu) , \tag{29}$$

where $S_{2n}^{(m)}$ are spherical harmonics and the assumption of even parity ensures that only even harmonics enter(29).

Expressed with respect to a line parallel to the particular wave normal which points into (λ_0, ν_0) we can write

$$f_0(\underline{p}) = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} f_{2n}^{(m)}(p) P_{2n}^{(m)}(\mu) \cos(m\varphi + \epsilon_{m,n}) \tag{30}$$

where the $\epsilon_{m,n}$ are suitably chosen constants and $P_{2n}^{(m)}(\mu)$ is the associated Legendre polynomial.

It follows that $\mu = 1$ when $\lambda = \lambda_0, \ \nu = \nu_0$ and then we have

$$P_{2n}^{(0)}(1) = 1 ; \qquad P_{2n}^{(m)}(1) = 0 \quad (m \neq 0) .$$

Equating harmonics of the same order in (29) and (30) we have

$$f_{2n}^{(0)}(p) = \sum_{m=0}^{2n} \psi_{2n}^{(m)}(p)\, S_{2n}^{(m)}(\lambda_0, \nu_0) ,$$

when $\lambda = \lambda_0,\ \nu = \nu_0.$

Thus

$$R_{2n}(p) = \sum_{m=0}^{2n} \psi_{2n}^{(m)}(p)\, S_{2n}^{(m)}(\lambda_0, \nu_0) .$$

Defining

$$\Psi_{2n}^{(m)} = \int_0^\infty \frac{\psi_{2n}^{(m)}(p)\; p^2 dp}{\sqrt{(1+p^2)}} ,$$

we have

$$r_{2n} = \sum_{m=0}^{2n} \Psi_{2n}^{(m)}\, S_{2n}^{(m)}(\lambda_0, \nu_0) .$$

Thus the requirement that instability be avoided can be written

$$\sum_{n=1}^{\infty} \sum_{m=0}^{2n} i_n \overline{\Psi}_{2n}^{(m)} S_{2n}^{(m)}(\lambda_0, \nu_0) \geqslant 0 \quad . \tag{31}$$

Now the average value of any spherical harmonic, of order unity or greater, over a sphere is zero. Therefore if the sum in (31) must not be negative for <u>any</u> values of λ_0 and ν_0 it must vanish for <u>all</u> λ_0, ν_0 . Since none of the i_n vanishes it follows that

$$\overline{\Psi}_{2n}^{(m)} = 0 \qquad (n \geqslant 1) .$$

Making use of (29) we see that this demands

$$\int_0^{\infty} \frac{p^2}{\sqrt{(1+p^2)}} f_0(p, \lambda_0, \nu_0)\, dp = F_0 , \text{ say} \tag{32}$$

and F_0 must be independent of λ_0 and ν_0 .

Denoting the solid angle element by $d\Omega$ it can be shown (Landau and Lifschitz, 1951) that

$$\frac{p^2 dp\, d\Omega}{\sqrt{(1+p^2)}}$$

is invariant under a Lorentz transformation.

Thus from (32) we can say that if electromagnetic instability is to be avoided then the number of relativistic particles moving into any given solid angle must be independent of the orientation of the solid angle.

When this is the case it can easily be seen that

$$M(0) = -8\pi r_0 + 8\pi/3 \left(J_0 - J_2/5\right)$$

for all direction of the wave normal. We also have

$$I_{yy}(0) = I_{zz}(0) \; ; \quad I_{yz}(0) = 0$$

in this case.

The requirement for avoiding electromagnetic instability can be made even stronger since we have that

$$\mathcal{J}_{yy} + \mathcal{J}_{zz} \leq g_y + g_z + I_{yy} + I_{zz} \tag{33}$$

with equality if, and only if, $I_y = 0 = I_z$.

Now at best the right hand side of (33) equals $-2g_x$ for all directions of the wave normal. Thus I_y and I_z must vanish for <u>all</u> directions of the wave normal in order that $\mathcal{J}_{yy} + \mathcal{J}_{zz}$ never be smaller than $-2g_x$ and thus that instability be avoided.

Both I_y and I_z are pure imaginary at $\beta = 0$ and hence at $p_x = 0$. Thus

$$\left(I_y , I_z \right) = i\pi \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} \left(p_y, p_z \right) \left. \frac{\partial f_0}{\partial p_x} \right|_{p_x = 0} dp_z \quad . \tag{34}$$

Changing to spherical momentum coordinates we have

$$\left(I_y, I_z \right) = -i\pi \int_0^{\infty} p\,dp \int_0^{2\pi} \left(\cos\varphi , \sin\varphi \right) \left. \frac{\partial f_0}{\partial \theta} \right|_{\theta = \pi/2} d\varphi \quad . \tag{35}$$

With f_0 given by (30) it can easily be seen that

$$\left[I_y(0), I_z(0) \right] = i\pi^2 \sum_{n=0}^{\infty} \left. \frac{d P_{2n}^{(1)}(\mu)}{d\mu} \right|_{\mu = 0} \int_0^{\infty} p f_{2n}^{(1)}(p) dp \; \left(\cos \epsilon_{1,n} \, , -\sin \epsilon_{1,n} \right) .$$

We again compare the two representations (29) and (30). We consider a particular wave normal through $\lambda = \lambda_0$, $\nu = \nu_0$ and let φ be measured in the plane containing the wave normal and the basic line.

Then, near $\lambda = \lambda_0$, $\nu = \nu_0$

$$\frac{\partial}{\partial \lambda} \equiv \frac{\partial}{\partial \theta} \quad , \varphi = 0 \; .$$

$$\frac{\partial}{\partial \theta} \equiv \operatorname{cosec} \lambda \, \frac{\partial}{\partial \nu} \, , \quad \varphi = \pi/2$$

Equating harmonics of equal order in (29) and (30) we see that

$$\sum_{m=0}^{2n} f_{2n}^{(m)}(p) \, P_{2n}^{(m)}(\mu) \cos(m\varphi + \epsilon_{m,n})$$
$$= \sum_{m=0}^{2n} \psi_{2n}^{(m)}(p) \, S_{2n}^{(m)}(\lambda, \nu) \tag{37}$$

Hence, near $\lambda = \lambda_0$, $\nu = \nu_0$ we have

$$\sum_{m=0}^{2n} \psi_{2n}^{(m)}(p) \left. \frac{\partial S_{2n}^{(m)}}{\partial \lambda} \right|_{\lambda_0, \nu_0} = f_{2n}^{(1)}(p) \left. \frac{d P_{2n}^{(1)}(\cos \theta)}{d\theta} \right|_{\theta=0} \cos \epsilon_{1,n} \; ,$$

$$\sum_{m=0}^{2n} \psi_{2n}^{(m)}(p) \operatorname{cosec} \lambda \left. \frac{\partial S_{2n}^{(m)}}{\partial \nu} \right|_{\lambda_0, \nu_0} = - f_{2n}^{(1)}(p) \left. \frac{d P_{2n}^{(1)}(\cos \theta)}{d\theta} \right|_{\theta=0} \sin \epsilon_{1,n} \; ,$$

where we have made use of the fact that

$$\frac{d P_{2n}^{(m)}(\cos\theta)}{d\theta} = 0 \qquad \text{at } \theta = 0 \text{ for } m \neq 1 .$$

Let

$$\chi_{2n}^{(m)} = \int_0^\infty \rho\, \psi_{2n}^{(m)}(\rho)\, d\rho .$$

Then

$$I_y(0) = \frac{\partial G}{\partial \lambda} , \quad I_z(0) = \operatorname{cosec}\lambda \frac{\partial G}{\partial \nu}$$

where

$$G = i\pi^2 \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \frac{\left(d P_{2n}^{(1)}(\mu)/d\mu\right)_{\mu=0}}{\left(d P_{2n}^{(1)}(\mu)/d\theta\right)_{\theta=0}} \chi_{2n}^{(m)} S_{2n}^{(m)}(\lambda, \nu)$$

We require that $I_y(0)$ and $I_z(0)$ vanish for all values of λ, ν . Hence a similar argument to that employed previously shows that

$$\chi_{2n}^{(m)} = 0 \qquad \text{for all } n, m .$$

Thus it follows that we must demand

$$\int_0^\infty p f_0(p, \lambda, \nu)\, dp = L_0, \text{ say} \tag{38}$$

where L_0 is a constant independent of λ, ν .

Thus if a relativistic plasma has a distribution function which satisfies (32) and (38) then there is no unstable electromagnetic disturbance with a real, non-zero wave number and zero phase velocity.

We can re-write (38) as

$$\int_0^\infty \frac{p^2 dp}{\sqrt{(1+p^2)}} \left[\frac{\sqrt{(1+p^2)}}{p} f_0(p, \lambda, \nu) \right] = L_0 , \tag{39}$$

and we note that $p(1+p^2)^{-1/2} = Vc^{-1}$ where V is the particle velocity. Thus (38) demands that the harmonic mean velocity of those particles moving in a given solid angle be independent of the spatial orientation of the solid angle. When (32) and (38) are satisfied we can show that no electromagnetic disturbance with a small imaginary β and real wave number can exist. To prove this we consider the values of J_{yy} , J_{zz} and J_{yz} near $\xi = 0$ $(\beta = i\xi)$.

Now

$$\mathcal{I}_{yy}(i\xi) = \mathcal{I}_{yy}(0) + \xi \left.\frac{\partial \mathcal{I}_{yy}}{\partial \xi}\right|_{\xi=0} + \frac{1}{2}\xi^2 \left.\frac{\partial^2 \mathcal{I}_{yy}}{\partial \xi^2}\right|_{\xi=0} + \cdots$$

and, since $I_y(0) = 0$,

$$\left.\frac{\partial \mathcal{I}_{yy}}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial I_{yy}}{\partial \xi}\right|_{\xi=0}$$

Likewise

$$\left.\frac{\partial^2 \mathcal{I}_{yy}}{\partial \xi^2}\right|_{\xi=0} = \left.\frac{\partial^2 I_{yy}}{\partial \xi^2}\right|_{\xi=0} - 2\, I_1^{-1}(0) \left(\frac{\partial I_y}{\partial \xi}\right)^2_{\xi=0} .$$

Making use of the fact that I_{yy}, I_1 are real and I_y is pure imaginary when β is pure imaginary it can be shown that

$$\left.\frac{\partial \mathcal{I}_{yy}}{\partial \xi}\right|_{\xi=0} = 0 .$$

and

$$\frac{1}{2}\left.\frac{\partial^2 \mathcal{J}_{yy}}{\partial \xi^2}\right|_{\xi=0} = -\int \frac{p_y^2\sqrt{(1+p^2)}}{p_x^3}\frac{\partial f_0}{\partial p_x} d^3p + \left[\int \frac{p_y\sqrt{(1+p^2)}}{p_x^2}\frac{\partial f_0}{\partial p_x} d^3p\right]^2 I_1^{-1}(0) .$$

(40)

Now

$$I_1(0) < 0$$

by definition.

Thus in order that $\partial^2 \mathcal{J}_{yy}/\partial \xi^2$ be positive it is both necessary and sufficient that

$$\int \frac{p_y^2\sqrt{(1+p^2)}}{p_x^3}\frac{\partial f_0}{\partial p_x} d^3p < 0 .$$

(41)

It is algebraically complicated, but quite straightforward, to show that (41) is obeyed. The method of proof consists of changing to spherical momentum coordinates, making use of (32) and expanding $f_0(\underset{\sim}{p})$ as in (29) and (30).

Consequently

$$\left.\frac{\partial^2 f_{yy}}{\partial \xi^2}\right|_{\xi=0} > 0 \quad ,$$

and thus

$$f_{yy}(i\xi) > -g_x \quad .$$

Likewise it can be shown that $f_{zz}(i\xi) > -g_x$ and $f_{yy}(i\xi) = f_{zz}(i\xi)$ to order ξ^2. It can also be shown that $f_{yz}(i\xi) = O(\xi^4)$. As a result, to order ξ^2, the dispersion relation becomes

$$\left[q + \frac{1}{2}\left(f_{yy} + f_{zz}\right)\right]^2 = 0 \quad . \tag{42}$$

Since $f_{yy}(i\xi) + f_{zz}(i\xi) > -2g_x$, it follows that the q corresponding to $\beta = i\xi \; (\xi > 0)$ is less than g_x. Consequently no real k exists for the given β value. Thus there are no unstable transverse waves which have an imaginary, but small, phase velocity and a positive wave number.

5. Conclusion.

Under the assumption that the plasma is stable against electrostatic waves, it has been shown that the plasma will support a class of growing transverse waves unless the number of protons moving into a given solid angle and their harmonic mean velocity are independent of the spatial orientation of the solid angle. This physical statement is identical to the result which obtains in the case of a non-relativistic, even parity, plasma (Kahn, 1962) except that the statement is now true for all plasmas both relativistic and non-relativistic. The mathematical formalism of the statement is changed in the relativistic case so that the conditions for stability remain invariant under a Lorentz transformation.

There are three interesting points worth noting.

Firstly, in principle it is possible to have pressure isotropy in the relativistic plasma and still have an electromagnetically unstable situation. In practice it is difficult to conceive of a physical situation where this will occur.

Secondly, we cannot state definitely that a relativistic plasma will be stable if its distribution function satisfies (32) and (38) since no account has been given of those values of k for which the phase velocity is different from zero. We can however, state that if the distribution function does not satisfy (32) and (38) then the plasma will be unstable.

Thirdly, we have considered only those waves for which $1 > \sigma >> |\beta|$. There still remains the class of slow electromagnetic waves for which $|\beta| \ll 1$ but $|\beta| \gtrsim \sigma$. No account has been given of these waves. Also we have not considered distribution functions which are not of even parity.

We have demonstrated that even when this relativistic plasma is stable against longitudinal waves this is no guarantee that the system is stable against any wave. We have not shown that similar conditions to (32) and (38) hold when an ambient magnetic field is introduced into the system.

Also no instability rates have been calculated. Thus even though the plasma is unstable to the transverse waves, the e-folding times may be so long that such waves are of no physical importance. The author feels, but so far has been unable to prove, that such waves probably have physically reasonable e-folding times. Thus they may be of importance in producing isotropy in the galactic cosmic ray gas in regions where the ambient galactic magnetic field is weak compared to its mean value.

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